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PROPIEDADES ALGEBRAICAS DE LAS MATRICES NATURALES

ALGEBRAIC PROPERTIES OF NATURAL MATRICES

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Resumen

Las matrices naturales, introducidas por el primer autor hace más de veinte años, son muy importantes en diversas áreas como la seguridad computacional, la criptografía, los sistemas dinámicos, las ecuaciones diferenciales, la biología y la física, entre otras investigaciones que actualmente se encuentran en curso. El objetivo principal de este trabajo es presentar nuevos resultados algebraicos en matrices naturales que fueron incluidos en la primera parte de la tesis de maestría en matemática aplicada del primer autor, supervisada por el segundo autor. Estos resultados son originales y corresponden a la tesis de maestría del primer autor, supervisada por el segundo autor. Reescribimos algunos resultados clásicos del álgebra lineal y abstracta a través de Matrices Naturales. Finalmente, establecemos hermosas y útiles relaciones entre Matrices Naturales y Cuadrados Mágicos, que son muy importantes en las matemáticas recreativas.

Palabras claves: Matrices naturales, cuadrados mágicos, algebra lineal.

Abstract

Natural matrices, introduced by the first author more than twenty years ago, are very important in several areas such as computational security, cryptography, dynamical systems, differential equations, biology and physics, among others researches that are currently in progress. The main aim of this paper is to present new algebraic results in natural matrices which were included in the first part of the master thesis in applied mathematics of the first author, supervised by the second author. These results are original and correspond to the master thesis of the first author, supervised by the second author. We rewrite some classical results of abstract and linear algebra through Natural Matrices. Finally, we establish beautiful and useful relationships between Natural Matrices and Magic Squares, which are very important in recreational mathematics.

Keywords: Natural matrices, magic squares, linear algebra.

1. Introduction

Mathematical sciences in terms of their human association are presented in our contemporaneity as applied mathematics and pure mathematics, it is usual to find academic programs labeled under the previous parameters in universities of various places in the world. The diversity of issues denoted and treated with due depth and rigor do not cease to be similar in the two aforementioned contexts, perhaps demonstrating and immersing oneself in the field of abstraction makes a difference between them, without forgetting the maturation of scientific knowledge because what at some moments at the historical level was considered abstract today is applied. Matrix theory is an example of interplaying between pure and applied mathematics.

Matrices are used in pure and in applied mathematics all time; in scientific computation, dynamical systems, differential equations, among others. New theories involving matrices have been developed recently, in where were introduced the concepts Natural Matrix, Successive Sums in Matrices, Pasting and Reversing in Matrices, Representation theory in algebraic groups of matrices, among others.

The main aim of this paper is the presentation of new scientific mathematical results concerning Natural Matrices in which are involved Linear Algebra (possibly the most important topic in mathematics for undergraduate and graduate students) and Magic Squares (topic of Recreational Mathematics). In particular, one aim is to prove that the trace of Natural Matrices are the Magic Constant in Magic Squares and the determinant of any natural matrix is zero whether the size is bigger than 2, that is, Natural Matrices are not invertible if and only if their size is greater than 2. We recall that as further applications of these results correspond to works in progress such as the analysis of differential equations, coding theory and computational security that includes cryptography, applications in biology, physics, and chemistry; through the theory of dynamical systems, among others, which are not considered here.

Our methodology is the same used by researchers in pure mathematics, which corresponds to the proposal of theorems, and after that theorems are proved. This methodology of research is apparently simple, but the starting point is very difficult because researchers should study mathematical theoretical background. After that, whether it is possible, are establishing conjectures (statements that should be proved or rejected) which in few case become in theorems once they are proved. In this paper, the methodology used here started with the analysis of references related with Natural Matrices, Linear Algebra and Magic Squares. Along this paper we refer to Magic Square as Magic Matrix.

2. Something about Natural Matrices and Magic Squares

In this section we start mentioning the state of art of Natural Matrices. The concept of Natural Matrix, joint with the concept Successive Sum, were introduced in 1998 by Ramírez and Gorrostola in [1]. Further developments concerning Natural Matrices correspond to [2, 3, 4, 5], where the authors used Pasting and Reversing in Natural Numbers [6, 7] and also Digital Root [8].

On the other hand, Pasting and Reversing is a theory developed by the first author in the same time as the second author joint with Gorrostola developed the theory of Natural Matrices.

3. Some Preliminaries about Natural Matrices, Linear Algebra and Magic Squares

An important branch of Mathematics is Linear Algebra. The aim of this work is to study from an algebraic point of view some properties of Natural Matrices. An standard reference to do an starting point is [9].

Natural matrices are matrices satisfying the following characteristics:

- All its components are natural numbers.
- Each component is unique within this arrangement (components are not repeated)
- Element 1 (first natural number) is located in the first row, first column
- The first row must be made up of the first k natural numbers. Where k corresponds to the number of columns of the matrix.
- The second row will have the successor of k, as the first component and its other components in the current order of the natural numbers, read from left to right from said component (k + 1), the other rows of the matrix are constructed in the same way, as an iterative process.

We can summarize the previous statements as follows: *A matrix* $N = [a_{ij}]_{nxm}$ *is said to be natural when each of its components can be written as*

$$a_{ij} = k(i-1) + j \tag{1}$$

Where

$$i = \left[\left|\frac{\lambda + k}{k}\right|\right] \qquad \textit{since } k \textit{ is not divisor of } \lambda, \textit{ or}$$

$$i = \frac{\lambda}{k} \qquad \textit{since } k \textit{ is divisor of } \lambda$$

$$j = \lambda - k\left[\left|\frac{\lambda + k}{k}\right|\right] \qquad \textit{since } k \textit{ is not divisor of } \lambda, \textit{ or}$$

$$j = k \qquad \textit{since } k \textit{ is not divisor of } \lambda$$

We recall that $|a|$ denotes the successive sum of a and that in case $m = 9$, the last two formulas become to

$$j = \lambda - 9\left[\left|\frac{\lambda+9}{9}\right|\right] \tag{2}$$

Thus, sometimes this formula is confused with digital root due to the results belongs to $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

It is clear that addition of natural matrices is not a natural matrix. Therefore, we can introduce the concept of Natural Matrix modified by a factor $t \in R$, to obtain the desired result. That is: $\alpha A + \beta B = \alpha N + \beta N = (\alpha + \beta)N$. It is because if A and B are natural matrices and $A + B$ exists, then $A = B = N$.

The following result is one contribution of this paper.

Theorem 1. The trace of a natural matrix modified by a factor $\alpha \in R$ is given by

$$Tr(A) = \alpha\left(\frac{n^3+n}{2}\right) \tag{3}$$

Proof. Let A be a natural matrix modified by a factor $\alpha \in R$. Thus $a_{ij} = n(i-1) + i$, for instance

$$Tra(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} (\alpha n(i-1) + \alpha i)$$

$$= \sum_{i=1}^{n} (\alpha n(i-1) + \alpha i)$$

$$= \sum_{i=1}^{n} \alpha n(i-1) + \sum_{i=1}^{n} \alpha i = \alpha n\left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right) + \alpha \sum_{i=1}^{n} i$$

$$= \alpha\left(n\frac{n^2+n}{2} - n\right) + \alpha\frac{n^2+n}{2} = \alpha n\left(\frac{n^2+n-2n}{2}\right) + \alpha\frac{n^2+n}{2}$$

$$= \alpha n\left(\frac{n^2-n}{2}\right) + \alpha\frac{n^2+n}{2}$$

$$\alpha\left(\frac{n^3-n^2+n^2+n}{2}\right) = \alpha\left(\frac{n^3+n}{2}\right)$$

Corollary 1. The trace of natural matrices is

$$Tr(A) = \frac{n^3+n}{2} \tag{4}$$

Proof. It is enough to consider α=1 in the previous theorem.

We want to know what happens with the determinant of natural matrices. To do this, we start considering the determinant of the natural matrix of size $2x2$ as follows:

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (3)(2) = -2 \tag{5}$$

Due to $|A| \neq 0$, we can obtain the inverse of the matrix A as follows:

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} \tag{6}$$

Now we consider the determinant of the following natural matrix:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$
$$= (1)(5)(9) + (2)(7)(6) + (4)(8)(3) - (4)(2)(9) - (6)(1)(8) - (7)(5)(3) = 0$$

In a general way it remains, which is presented in the following theorem.

Theorem 2. Let A be a natural matrix of size $n \times n$, such that $n \geq 3$. Then $|A| = 0$.

Proof. We observe that at least one row is reduced to zero by means of elementary operations between rows. In particular, the elementary operation used to reduce the row n to zero is the addition of the row 1 with row n minus the addition of the row 2 minus the row $n-1$, that is $f_n \rightarrow (f_1 + f_n) - (f_2 + f_{n-1})$.

The following theorem is related with the rank of a natural matrix.

Theorem 3. Let A be a natural matrix of size $n \times n$, such that $n \geq 3$. Then $rank(A) = 2$.

Proof. It follows from the number of rows that are not reduced to zero. By direct computation we can see that elementary operations are made by pairs of rows as in the previous theorem.

4. Contribution to magic squares

A magic square, which we call magic matrix, is a matrix in where addition by rows, columns and diagonal is constant. Such constant is called *magic constant*.

Do not exist a magic matrix of size $2x2$. The magic matrices appear since 3x3 matrices.

The following are the results relating magic matrices with natural matrices.

Theorem 4. Do not exist linear transformations that become natural matrices to magic matrices and vice versa.

Proof. It follows directly by Theorem 2 and Theorem 3.

Theorem 5. The magic constant of the square matrix of size nxn is exactly the trace of the natural matrix of size nxn.

Proof. It follows directly because the magic constant is $\frac{n(n^2+1)}{2}$, which corresponds to the trace of the natural matrix of size nxn.

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